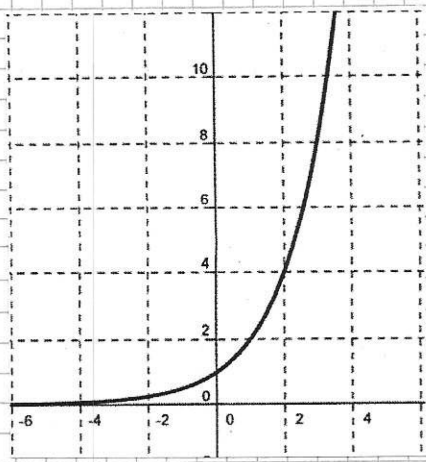
Chapitre 6. Fonctions exponentielles et logarithmes – Corrigé

Exercice 1

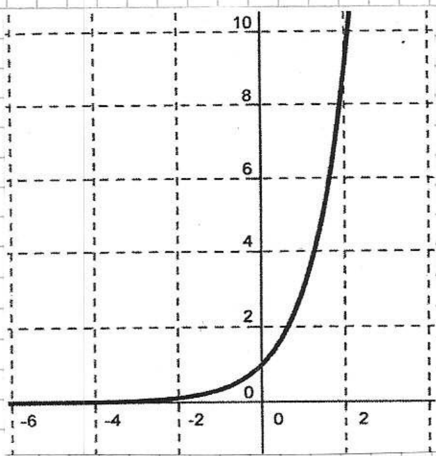
1. $2^x = 8 \Rightarrow \underline{x=3}$ car $2^3 = 8$.
2. $7^x = 1 \Rightarrow \underline{x=0}$ car $7^0 = 1$.
3. $3^x - 81 = 0 \Rightarrow 3^x = 81 \Rightarrow \underline{x=4}$ car $3^4 = 81$.
4. $2^x = \frac{1}{\sqrt{2}} = \frac{1}{2^{1/2}} = 2^{-1/2} \Rightarrow \underline{x = -\frac{1}{2}}$.
5. $3^x = \sqrt[3]{9} = \sqrt[3]{3^2} = (3^2)^{\frac{1}{3}} = 3^{2\cdot\frac{1}{3}} = 3^{\frac{2}{3}} \Rightarrow \underline{x = \frac{2}{3}}$.
6. $2^x = 2\cdot\sqrt[5]{16} = 2\cdot\sqrt[5]{2^4} = 2\cdot(2^4)^{\frac{1}{5}} = 2\cdot 2^{4\cdot\frac{1}{5}} = 2^1\cdot 2^{\frac{4}{5}} = 2^{1+\frac{4}{5}} = 2^{\frac{9}{5}} \Rightarrow \underline{x = \frac{9}{5}}$.
7. $10^x = 0{,}001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3} \Rightarrow \underline{x=-3}$.
8. $\left(\frac{3}{4}\right)^x = \frac{4}{3} \Rightarrow \underline{x=-1}$ car $\left(\frac{3}{4}\right)^{-1} = \frac{1}{3/4} = 1 : \frac{3}{4} = 1\cdot\frac{4}{3} = \frac{4}{3}$.
9. $4^x - 32 = 0 \Rightarrow 4^x = 32 \Rightarrow (2^2)^x = 2^5 \Rightarrow 2^{2x} = 2^5 \Rightarrow 2x = 5 \Rightarrow \underline{x = \frac{5}{2}}$.
10. $4^x = \sqrt{32} \Rightarrow 4^x = 32^{\frac{1}{2}} \Rightarrow (2^2)^x = (2^5)^{\frac{1}{2}} \Rightarrow 2^{2x} = 2^{\frac{5}{2}} \Rightarrow 2x = \frac{5}{2} \Rightarrow \underline{x = \frac{5}{4}}$.
11. $3^x = 9^{x+1} \Rightarrow 3^x = (3^2)^{x+1} \Rightarrow 3^x = 3^{2(x+1)} \Rightarrow x = 2(x+1) \Rightarrow x = 2x+2 \Rightarrow -x = 2$
$\Rightarrow \underline{x=-2}$.
12. $2^{\sqrt{x}} = 16 \Rightarrow 2^{\sqrt{x}} = 2^4 \Rightarrow \sqrt{x} = 4 \Rightarrow \underline{x=16}$.
13. $3^{3^x} = 3 = 3^1 \Rightarrow 3^x = 1 \Rightarrow \underline{x=0}$.
14. $4^{x^2} = 2^{5x-2} \Rightarrow (2^2)^{x^2} = 2^{5x-2} \Rightarrow 2^{2x^2} = 2^{5x-2} \Rightarrow 2x^2 = 5x-2 \Rightarrow 2x^2 - 5x + 2 = 0$;
$a=2,\ b=-5,\ c=2 \Rightarrow b^2 - 4ac = (-5)^2 - 4\cdot2\cdot2 = 25-16 = 9 > 0$
$\Rightarrow x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{5+\sqrt{9}}{2\cdot2} = \frac{5+3}{4} = \underline{2}$ et $x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{5-\sqrt{9}}{2\cdot2} = \frac{5-3}{4} = \underline{\frac{1}{2}}$.

Exercice 2

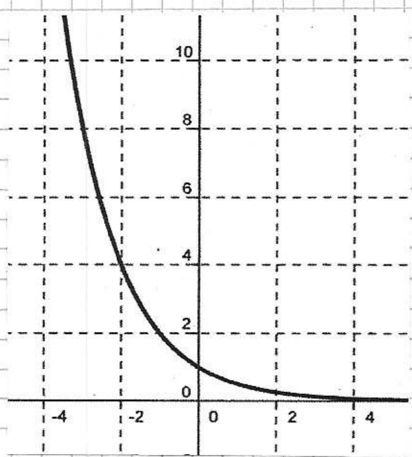
$f(x) = 2^x$:



$g(x) = 3^x$:



$h(x) = 2^{-x}$:



Exercice 3

De manière générale, $\log_a x = y$ signifie que $x = a^y$.

De plus $\log_a b = \frac{\log b}{\log a}$, ce qui est faisable à la calculatrice.

1. $\log_4 x = -1{,}5 \Rightarrow x = 4^{-1,5} = \underline{0{,}125}$.

2. $\log_3 250 = x \Rightarrow x = \frac{\log 250}{\log 3} \cong \underline{5{,}026}$.

3. $\log_x 1296 = 4 \Rightarrow 1296 = x^4 \Rightarrow x = \sqrt[4]{1296} = \underline{6}$.

4. $\log_5 100 = x \Rightarrow x = \frac{\log 100}{\log 5} \cong \underline{2{,}861}$.

5. $\log_x 2 = 0{,}3 \Rightarrow 2 = x^{0,3} \Rightarrow x = \sqrt[0,3]{2} \cong \underline{10{,}079}$.

6. $\log_3 x = 2{,}465 \Rightarrow x = 3^{2,465} \cong \underline{15{,}0004}$.

7. $\log_{81} x = 2 \Rightarrow x = 81^2 = \underline{6561}$.

8. $\log_2 81 = x \Rightarrow x = \frac{\log 81}{\log 2} \cong \underline{6{,}340}$.

9. $\log_x 81 = 2 \Rightarrow 81 = x^2 \Rightarrow \underline{x = 9}$.

Exercice 4

On a $C_0 = 100'000.-$, $t = 2\% = 0{,}02$.

1. $n = 6$ mois $= 0{,}5$ année $\Rightarrow C_{1/2} = 100'000 \cdot (1+0{,}02)^{1/2} \cong \underline{100'995{,}05}$.
2. $n = 6$ années $\Rightarrow C_6 = 100'000 \cdot (1+0{,}02)^6 \cong \underline{112'616{,}24}$.
3. $n = 20$ années $\Rightarrow C_{20} = 100'000 \cdot (1+0{,}02)^{20} \cong \underline{148'594{,}74}$.

Exercice 5

On a $C_0 = 100'000$, $n = 10$ années et $C_{10} = 134'391{,}65$

$$\Rightarrow 134'391{,}65 = 100'000 \cdot (1+t)^{10} \quad \Big| \; : 100'000$$

$$1{,}3439165 = (1+t)^{10} \quad \Big| \; \sqrt[10]{}$$

$$1{,}03 = 1+t \quad \Big| \; -1$$

$$0{,}03 = t$$

$\Rightarrow$ le taux était de $0{,}03 = \underline{3\%}$.

Exercice 6

On a $t = 2\% = 0{,}02$, $n = 10$ années et $C_{10} = 121'899{,}45$

$\Rightarrow 121'899{,}45 = C_0 \cdot (1+0{,}02)^{10}$ | Calculs

$121'899{,}45 = C_0 \cdot 1{,}21899442$ | $: 1{,}21899442$

$100'000 = C_0$

$\Rightarrow$ la somme initiale était de <u>100'000.–</u>

Exercice 7

On a $t = 2\% = 0{,}02$, $C_0 = 100'000$.

1. $C_n = 102'000 \Rightarrow 102'000 = 100'000\,(1+0{,}02)^n$ | $: 100'000$

$1{,}02 = 1{,}02^n$

$\Rightarrow n = 1$

$\Rightarrow$ après <u>1 an.</u>

2. $C_n = 134'587 \Rightarrow 134'587 = 100'000\,(1+0{,}02)^n$ | $: 100'000$

$1{,}34587 = 1{,}02^n$

$\Rightarrow n = \dfrac{\log 1{,}34587}{\log 1{,}02} \cong 15$

$$\boxed{a = b^x \Rightarrow x = \frac{\log a}{\log b}}$$

$\Rightarrow$ après <u>15 ans.</u>

3. $C_n = 100'995 \Rightarrow 100'995 = 100'000 \cdot (1+0{,}02)^n$ | $: 100'000$

$1{,}00995 = 1{,}02^n$

$\Rightarrow n = \dfrac{\log(1{,}00995)}{\log(1{,}02)} \cong 0{,}5$

$a = b^x \Rightarrow x = \dfrac{\log a}{\log b}$

$\Rightarrow$ après <u>0,5 années = 6 mois.</u>

Exercice 8

On a $V_0 = 100'000$, $t = 10\% = 0{,}1$ et $n = 10$ ans

$\Rightarrow V_{10} = 100'000 \cdot (1-0{,}1)^{10} = \underline{34'867{,}84}$.

Exercice 9

On a $V_0 = 100'000$, $t = 12\% = 0{,}12$ et $V_n = \dfrac{100'000}{2} = 50'000$

$\Rightarrow 50'000 = 100'000\,(1-0{,}12)^n \Rightarrow 0{,}5 = 0{,}88^n \Rightarrow n = \dfrac{\log(0{,}5)}{\log(0{,}88)} \cong 5{,}4$

$\Rightarrow$ Dans <u>5,4 années = 5 ans 5 mois.</u>

Exercice 10

On a $V_0 = 60'000$, $n = 10$ ans et $V_n = 30'000$

$$\Rightarrow 30'000 = 60'000\,(1-t)^{10} \quad \Big| \; : 60'000$$
$$0{,}5 = (1-t)^{10} \quad \Big| \; \sqrt[10]{}$$
$$0{,}933 = 1-t \quad \Big| \; -1$$
$$-0{,}067 = -t \quad \Big| \; \cdot(-1)$$
$$0{,}067 = t$$

$\Rightarrow$ le taux doit être de $0{,}067 = \underline{6{,}7\,\%}$.

Exercice 11

On a $V_0 = 100'000$ et $t = 5\% = 0{,}05$.

1. $n = 10$ ans $\Rightarrow V_{10} = 100'000 \cdot (1-0{,}05)^{10} \cong \underline{59'873{,}69}$.

2. $V_n = \frac{1}{4} V_0 = \frac{1}{4} \cdot 100'000 = 25000 \Rightarrow$

$$25000 = 100000\,(1-0{,}05)^n \quad \Big| \; : 100'000$$
$$0{,}25 = 0{,}95^n \quad \Big| \; a = b^x \Rightarrow x = \frac{\log a}{\log b}$$
$$\Rightarrow n = \frac{\log 0{,}25}{\log 0{,}95} \cong \underline{27 \text{ ans.}}$$

3. $V_0 = 100'000$, $n = 20$ et $V_{20} = 50'000 \Rightarrow$

$$50'000 = 100'000\,(1-t)^{20} \quad \Big| \; : 100'000$$
$$0{,}5 = (1-t)^{20} \quad \Big| \; \sqrt[20]{}$$
$$0{,}966 = 1-t \quad \Big| \; -1$$
$$-0{,}034 = -t \quad \Big| \; \cdot(-1)$$
$$t = 0{,}034 = \underline{3{,}4\%}$$

Exercice 12

On a $P(t) = P_0 e^{0,012t}$ où $P_0 = 25'000$ et t est en années.

1. Avec $t = 10$ ans, on a $P(10) = 25'000 \cdot e^{0,012 \cdot 10} \cong \underline{28'187}$.

2. $P(t) = 2 \cdot 25'000 = 50'000 \Rightarrow$

$$\begin{aligned} 50'000 &= 25'000 \cdot e^{0,012 \cdot t} && \Big| \; : 25'000 \\ 2 &= e^{0,012 \cdot t} && \Big| \; \ln(\quad) \text{ (opération inverse de } e^{\cdots}) \\ \ln(2) &= 0,012 \cdot t && \\ 0,693 &= 0,012 \cdot t && \Big| \; : 0,012 \\ 57,76 &= t && \end{aligned}$$

$\Rightarrow$ dans $\underline{57,76 \text{ années}}$.

Exercice 13

On a $P(t) = P_0 e^{\alpha t}$.

1. Europe: $P_0 = P$ en $1970 = \underline{495,5}$.
En 1990, $t = 20$ ans et $P(20) = 553,4$
$\Rightarrow P_0 e^{\alpha \cdot 20} = 553,4 \Rightarrow 495,5 e^{20\alpha} = 553,4$
$\Rightarrow e^{20\alpha} = 1,11685 \Rightarrow 20\alpha = \ln(1,11685) = 0,1105$
$\Rightarrow \underline{\alpha = 0,005526}$.

2. Amérique latine: $P_0 = P$ en $1970 = \underline{285,1}$.
En 1990, $t = 20$ ans et $P(20) = 448,1$
$\Rightarrow P_0 e^{\alpha \cdot 20} = 448,1 \Rightarrow 285,1 e^{20\alpha} = 448,1$
$\Rightarrow e^{20\alpha} = 1,5717 \Rightarrow 20\alpha = \ln(1,5717) = 0,4522$
$\Rightarrow \underline{\alpha = 0,022609}$.

3. Europe: $P(t) = 495,5 e^{0,005526 \cdot t}$
Amérique latine: $P(t) = 285,1 e^{0,022609 \cdot t}$
Europe = Amérique latine $\Rightarrow 495,5 e^{0,005526 \cdot t} = 285,1 e^{0,022609 \cdot t}$
$\Rightarrow \dfrac{495,5}{285,1} = \dfrac{e^{0,022609 \cdot t}}{e^{0,005526 \cdot t}} \Rightarrow 1,738 = e^{0,022609 \cdot t - 0,005526 \cdot t}$
$\Rightarrow 1,738 = e^{0,017083 \cdot t} \Rightarrow 0,017083 \cdot t = \ln(1,738) = 0,5527$
$\Rightarrow t \cong 32,36$.
Donc après 32 ans, soit en $1970 + 32 = \underline{2002}$.

Exercice 14

On a $V(t) = V_0 e^{0,03t}$

1. On a $V_0 = 400'000$ et $t = 7$ ans $\Rightarrow V(t) = 400'000 \cdot e^{0,03 \cdot 7} \cong \underline{493'471,20}$.

2. $V(t) = 650'000 \Rightarrow 650'000 = 400'000\, e^{0,03t} \Rightarrow 1,625 = e^{0,03t}$
$\Rightarrow 0,03t = \ln(1,625) \cong 0,4855 \Rightarrow t \cong 16,18 \Rightarrow \underline{\sim 16 \text{ ans}}$.

Exercice 15

On a $V_0 = 35'000$, $t = 8\% = 0,08$ et $V_n = 15'000$
$V_n = V_0 \cdot (1-t)^n \Rightarrow 15'000 = 35'000\,(1-0,08)^n \Rightarrow 0,42857 = 0,92^n$
$\Rightarrow n = \dfrac{\log(0,42857)}{\log(0,92)} = \underline{10,16 \text{ ans} = 10 \text{ ans } 2 \text{ mois}}$.

Exercice 16

On a $A = 100(1 - 0,9^t)$, t en minutes.
$A = 50$ mg $\Rightarrow 50 = 100(1-0,9^t) \Rightarrow 0,5 = 1 - 0,9^t \Rightarrow -0,5 = -0,9^t$
$\Rightarrow 0,9^t = 0,5 \Rightarrow t = \dfrac{\log 0,5}{\log 0,9} \cong \underline{6,58 \text{ minutes} \cong 6 \text{ min } 35 \text{ sec}}$.

Exercice 17

On a $A = 10 \cdot 0,8^t$, t en heures.

1. $A = 2$ mg $\Rightarrow 2 = 10 \cdot 0,8^t \Rightarrow 0,2 = 0,8^t \Rightarrow t = \dfrac{\log 0,2}{\log 0,8} \cong \underline{7,213 \text{ heures}}$
$\underline{\cong 7\text{h } 12\text{min } 45\text{s}}$.

2. La demi-vie est le temps où il n'y a plus que la moitié de la substance présente.
On doit donc avoir $A = \frac{10}{2} = 5$ mg
$\Rightarrow 5 = 10 \cdot 0,8^t = 0,5 = 0,8^t \Rightarrow t = \dfrac{\log(0,5)}{\log(0,8)} \cong \underline{3,106 \text{ h} \cong 3\text{h } 6\text{min } 23\text{s}}$.

Exercice 18

On a $Q = 2 \cdot 3^t$, Q en milliers de bactéries et t en heures.

1. $t = 0 \Rightarrow Q = 2 \cdot 3^0 = 2 \cdot 1 = 2$ milliers $=$ 2000 bactéries.

2. $t = 10\,\text{min} = \frac{1}{6}\,\text{h} \Rightarrow Q = 2 \cdot 3^{\frac{1}{6}} \cong 2{,}40187$ milliers $\cong$ 2402 bactéries.
   $t = 30\,\text{min} = 0{,}5\,\text{h} \Rightarrow Q = 2 \cdot 3^{0{,}5} \cong 3{,}4641$ milliers $\cong$ 3464 bactéries.
   $t = 1\,\text{h} \Rightarrow Q = 2 \cdot 3^1 = 6 =$ 6000 bactéries.

Exercice 19

On a $h = \dfrac{36}{1+200e^{-0{,}2t}}$, où t est en années et h en mètres.

1. $t = 10$ ans $\Rightarrow h = \dfrac{36}{1+200\,e^{-0{,}2\cdot 10}} \cong \dfrac{36}{28{,}067} \cong 1{,}28$ m.

2. $h = 15$ m $\Rightarrow 15 = \dfrac{36}{1+200e^{-0{,}2t}} = 15\left(1+200e^{-0{,}2t}\right) = 36$
   $\Rightarrow 1+200e^{-0{,}2t} = 2{,}4 \Rightarrow 200e^{-0{,}2t} = 1{,}4 \Rightarrow e^{-0{,}2t} = 0{,}007$
   $\Rightarrow -0{,}2t = \ln(0{,}007) \cong -4{,}96 \Rightarrow t = 24{,}8$ ans.

Exercice 20

On a $N = 20{,}2e^{0{,}041t}$, N en millions d'habitants et t en années à partir de 1985 ($t = 0$ en 1985).

Si $t = 0$, on a $N = 20{,}2\,e^{0{,}041\cdot 0} = 20{,}2 \cdot e^0 = 20{,}2 \cdot 1 = 20{,}2$.

La population double : $2N = 2 \cdot 20{,}2 = 40{,}4$

$\Rightarrow 40{,}4 = 20{,}2\,e^{0{,}041t} \Rightarrow 2 = e^{0{,}041t} \Rightarrow 0{,}041t = \ln 2 = 0{,}693$

$\Rightarrow t \cong 16{,}9$ ans.